temperature source. Temperature control proved to be difficult, and direct electrical heating was used for later tests. The stainless steel substrate was heated by passing current through it.

The sample preparation and apparatus is the same as that used for the high-temperature calorimetric hemispherical emittance test. Thermocouples are placed along the sample, and a voltage is applied to the ends of the sample. The current is adjusted to give the desired temperature profile. All of these simulated plume heating tests were conducted within a vacuum of 10^{-6} torr or better.

Property measurements were made ex situ following the simulated plume heating tests and prior to subsequent tests.

Following the plume heating test, the samples were subjected to ultraviolet irradiation in vacuum. The measurement of spectral reflectance in the solar region was made in situ. This in situ apparatus is described elsewhere.[6] The design is such that any ultraviolet-induced reflectance changes can be measured without returning the sample to air. After the environmental tests had been conducted, the final property measurements were made.

Results

The results of the tests indicated that, for the low α/ϵ materials studied, there appeared to be no significant synergism between the plume heating and the ultraviolet irradiation. These results are shown in Table 1. Although not shown in the table, the polyimide film exhibited stable properties after exposure to 650°K in vacuo. Although not useable for this specific application, the material may be suitable under less severe conditions.

Table 2 shows the results for several high-temperature surfaces. For these materials, the solar absorptance is not important, and only the stability to the high-temperature exposure is significant. It is a characteristic of some alloys that a permanent change in emittance results from particular temperature exposure. This is shown in the first three samples in Table 2. The emittance of Haynes 25 decreases after heating to 1365°K where adsorbed surface gases are driven off. Above 1475°K, grain growth and/or other physical changes result in an increased emittance. The emittance of Columbium C103 increases slightly after 1700°K exposure, but decreases significantly after 1920°K exposure. The data for several high emittance coatings developed at TRW Systems are also given. These have been used where stable high emittance at high temperature is required.

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A Comparison of Approximate Analytical Techniques in Heat Conduction

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Introduction

A NUMBER of approximate analytical methods have been used in heat flow analysis.[1–9] Among these, the method of collocation,[2] Galerkin's method,[3] and the method of moments[2,4] (which includes the so-called heat balance integral)[5] are all classified as weighted residual techniques. A somewhat different technique, employing variational principles, was formulated by Biot over a decade ago[6] and has since been applied to nonlinear heat-conduction problems by Lardner.[7,8] A variational treatment of the heat equation was also developed by Washizu.[9] None of these approximate methods is restricted to heat-conduction problems alone. This Note compares the results of using each of them to solve two simple one-dimensional heat-conduction problems, in which a semi-infinite solid, initially at a uniform temperature, is suddenly exposed to 1) a constant surface temperature or 2) a constant surface flux, as the boundary condition.

Weighted Residual Methods

The one-dimensional heat-conduction equation for the case of constant material properties is

$$L(\theta) = \alpha\partial^2\theta/\partial x^2 - \partial\theta/\partial t = 0 \quad (1)$$

where α is the thermal diffusivity defined as the ratio of the thermal conductivity to the product of density and specific heat ($k/\rho c_p$), θ is the temperature above some reference temperature and $L(\theta)$ represents the operator for the heat equation. An approximate solution, θ_η, when substituted into Eq. (1) results in a residual or error term ϵ_η,

$$L(\theta_\eta) = \epsilon_\eta \quad (2)$$

Suppose the residual is multiplied by a weighting function W_j, and averaged over the space ordinate. If the average is set equal to zero, the following form is obtained:

$$\int_x W_j L(\theta_\eta)dx = 0, j = 1,2, \ldots, \eta \quad (3)$$

Thus, the solution is forced to satisfy Eq. (3) such that the operator on the approximate solution is orthogonal to the weighting factor over all values of the space ordinate. Generally, the form of θ_η is selected so that the boundary conditions are automatically satisfied, although this is not absolutely necessary. The form of θ_η will contain η arbitrary parameters q_η. These are evaluated by choosing a weighting function and applying Eq. (3). The result will be η equa-

Table 1 Comparison of results: constant surface temperature; assumed profile, $\theta = \theta_s(1 - x/\delta)^2$; result, $\delta = C(\alpha t)^{1/2}$

Approximate analytical method	C	$x/(\alpha t)^{1/2}$, where $\theta/\theta_s = 0.67$	Difference, %
Collocation at $x = \delta/2$	2.83	0.515	−14.2
Collocation at centroid, $x = \delta/4$	3.265	0.594	−1.0
Heat balance integral	3.46	0.630	+5.0
Galerkin method	3.17	0.577	−3.8
Biot method	3.36	0.611	+1.8
Exact solution	...	0.600	...

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tions which can be solved simultaneously for the unknown parameters.

The weighting factor for collocation is the Dirac delta function which reduces Eq. (3) to the solution of the heat equation at each collocation point. For this reason, collocation is analogous to curve-fitting with the assumed solution satisfying the heat equation at all collocation points. The method of moments uses x^j as the weighting function, and for the special case $j = 0$ results in the heat balance integral formulation. The solution by the heat balance method satisfies the heat equation on the average. Finally, application of the Galerkin method involves orthogonalizing the heat equation with a weighting function based on the assumed temperature profile $\partial\theta_\eta/\partial q_j$.

Biot's Variational Method

The variational principle is developed through the use of a heat flow vector H, whose time rate of change is the heat flux across an area normal to H.[6] The mathematical statement of the variational principle is written as

$$\delta V + \delta D = -\theta\delta H \tag{4}$$

where

$$H = -\int_x\int_\theta \rho c_p d\theta dx \tag{5}$$

$$V = \int_x\int_\theta \rho c_p \theta d\theta dx \tag{6}$$

$$D = \int_x (1/2k)(\dot{H})^2 dx \tag{7}$$

For arbitrary variations in the heat flowfield, it can be shown that Eq. (4) is equivalent to the heat-conduction equation. The variational principle can be expressed in terms of generalized coordinates $q_\eta(t)$, in a Lagrangian form[6]:

$$\partial V/\partial q_i + \partial D/\partial \dot{q}_i = Q_i \tag{8}$$

where

$$Q_i = (\theta\partial H/\partial q_i)_{x=0} \tag{9}$$

As with the residual methods, the solution depends upon assuming some form for the temperature profile in terms of η generalized coordinates, and these are determined by using Eqs. (5–9) of the variational formulation.

When the boundary conditions involve the temperature explicitly, the foregoing formulation is sufficient to determine the generalized coordinates and therefore the solution is determined directly. However, when the boundary condition is specified in terms of the flux, an additional equation should be used to define the over-all energy balance at the surface. For the one-dimensional case this takes the form

$$\dot{H}_n = F \tag{10}$$

where F is the prescribed heat flux into the body and n is the normal vector to the surface, the direction into the body taken as positive. Note that the variational principle as stated in the foregoing equations does not introduce any boundary conditions; that is, natural boundary conditions obtained from conventional variational principles. Emery[10] has demonstrated that slightly different results will be obtained when the boundary condition on the temperature gradient at the surface is used instead of Eq. (10). However, this Note follows the approach of Biot[6] and Lardner[8] and uses Eq. (10). The variational method considers $(\eta - 1)$ of the generalized coordinates as independent coordinates to be determined from the variational principle with the ηth coordinate determined from the over-all energy balance expressed through Eq. (10).

Each approximate method reduces the heat equation to one or more first-order ordinary differential equations. Since the material properties are taken as constant in the following examples, these equations can be solved in closed form. Only the solutions resulting from the use of each method are discussed herein; a more complete analysis and a comprehensive bibliography are contained in Ref. 11.

Solutions and Discussion

For each problem a parabolic temperature profile is assumed in the form

$$\theta = q_2(1 - x/q_1)^2 \tag{11}$$

where for the constant surface temperature boundary condition $q_2 = \theta_s$, with $q_1 = \delta$ the single generalized coordinate. For the case of a constant flux boundary condition, both the surface temperature θ_s and the penetration depth of the temperature rise δ enter the analysis as generalized coordinates. The objective of each approximate technique is to determine the best values of q_i consistent with that formulation.

Constant surface temperature boundary condition

The solutions resulting from each approximate method for the assumed temperature profile are summarized in Table 1. The penetration depth has the form

$$\delta = C(\alpha t)^{1/2} \tag{12}$$

where the value of C is derived from the particular method being used. In addition, the value of $x/(\alpha t)^{1/2}$ where the temperature has dropped to 67% of the surface temperature is compared to the value computed from the exact solution;

$$\theta = \theta_s \operatorname{erfc}[x/2(\alpha t)^{1/2}] \tag{13}$$

where $\operatorname{erfc}\xi$ is the complementary error function of ξ.

It is evident that the solution given by collocation is not unique, the result depending on the locations of the collocation point. In obtaining the second entry in Table 1 for the collocation method, Eq. (3) was collocated such that the more active regions of the profile would have a greater in-

Table 2 Comparison of results: constant surface flux; assumed profile, $\theta = \theta_s(1 - x/\delta)^2$; result, $\delta = C_1(\alpha t)^{1/2}$ and $\theta_s = C_2(F/k)(\alpha t)^{1/2}$

Approximate analytical method	C_1	C_2	Difference, %	Remarks
Collocation	2.065	1.0325	−8.4	At centroid $x = \delta/4$
Heat balance integral	2.45	1.225	+8.6	Parabolic temperature profile
Galerkin method	2.39	1.195	+5.9	Parabolic, δ-independent
	2.11	1.055	−6.5	Parabolic, θ_s-independent
Biot method	2.81	1.068	−5.3	Parabolic, flux condition
	2.677	1.120	−0.7	Parabolic, δ-independent
	2.59	1.158	+2.6	Parabolic, θ_s-independent
Heat balance integral[a]	3.45	1.15	+2.0	Cubic temperature profile
Biot method[a]	3.56	1.123	−0.4	Cubic, δ-independent
	0.866	1.152	+2.1	Exponential, δ-independent
Exact solution	...	1.128	...	...

[a] Solutions are based on profiles indicated in the remark column.[7]

fluence on the solution. Both the heat balance integral method and the Galerkin method result in one solution. Of the three methods which lead to a unique value for the penetration depth under the assumption of the parabolic temperature profile, the Biot method results in the best agreement with the exact solution.

Constant flux boundary condition

The results for the flux boundary condition are compared in Table 2, where the penetration depth and surface temperature are, respectively,

$$\delta = C_1(\alpha t)^{1/2},\ \theta_s = C_2(F/k)(\alpha t)^{1/2} \tag{14}$$

In obtaining the entries for the collocation and heat balance methods, θ_s was related to δ directly through the flux condition

$$F = -k(\partial\theta/\partial x) \tag{15}$$

The solution by Galerkin's method was determined using both θ_s and δ separately, as weighting variables and therefore two different values of C_1 and C_2 are listed in Table 2. Each of the residual techniques transformed the heat equation into a first-order ordinary differential equation which was solved for the unknown generalized coordinate. However, the Biot method resulted in two ordinary differential equations: one given by the Lagrangian equation, Eq. (8), and one from the constraint equation, Eq. (10). As with the Galerkin technique, the solution was obtained by using either δ or θ_s as the independent generalized coordinate. Note, however, a somewhat different approach can be taken in the variational formulation if instead of Eq. (11), the profile is taken as

$$\theta = A(t)(1 - x/\delta)^2 \tag{16}$$

where $A(t)$ is a function of the heat flux. By using the constraint equation directly, $\dot{H} = F$ at $x = 0$, θ_s is eliminated from the analysis, and a single differential equation for δ must be solved. This result is indicated as the "flux condition" in Table 2. Also shown in Table 2 are the numerical values which result when temperature profiles other than parabolic are assumed.[7] Table 2 shows that the Biot method gives the best agreement with the exact solution for surface temperature.

Conclusions

For the two simple problems discussed in this Note, each approximate method involved about the same amount of analytical effort. However, for nonlinear problems in which the thermal properties are temperature-dependent and the boundary conditions involve both convection and radiation which may change with time, the Biot method is probably the most flexible to apply. In this regard, Ref. (11) presents results based on the Biot formulation for the one-dimensional heat conduction in flat plates, which compare favorably with those from a finite-difference technique.

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Tube-Wall Fin Effects in Spacecraft Radiators

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Nomenclature

h = convection heat-transfer coefficient, Btu/hr-ft²-°F
k = thermal conductivity, Btu/hr-ft-°F
L_f = fin length, ft
Nu = $2R\int_0^\pi q_w(\theta)d\theta/(\pi k_f)$, dimensionless
Q = heat-transfer rate, per unit length of radiator, Btu/hr-ft
$q_w(\theta)$ = heat flux to wall, Btu/hr-ft²
R = tube radius, ft
r = radial coordinate, made dimensionless with R
T = temperature, °F
$\bar{T}_f$ = $\left(\int_0^\pi\int_0^R T_f w r dr d\theta\right)\Big/\left(\int_0^\pi\int_0^R w r dr d\theta\right)$
t = tube-wall thickness, ft
w = fluid velocity, made dimensionless with $\bar{w}$
$\bar{w}$ = average fluid velocity, ft/hr
z = axial coordinate, made dimensionless with $\alpha/(\bar{w}R^2)$
α = thermal diffusivity of fluid, ft²/hr
θ = circumferential angle, deg
ν = $k_f R/(k_w t)$
η = fin effectiveness

Subscripts

f,w = fluid and wall, respectively
w_1,w_2 = tube wall region $\theta \leq \theta_1$ and $\theta_1 < \theta$, respectively
0 = values calculated with isothermal wall

Introduction

PREVIOUS analyses of spacecraft radiator performances[1–5] are based in part on the assumption that heat transfer between the fluid (at T_f) and the tube wall (at T_w) is governed by $Q = h(2R)(\bar{T}_f - T_w)$ where h is calculated from theories derived on the assumption that T_w is uniform in the circumferential direction. This Note presents 1) a procedure for including the effects of nonuniform wall temperatures and 2) a complete solution for the extreme case of nonuniform wall temperature in laminar flow.

Figure 1 represents a cross-section of one tube, or fluid passage, and its associated fin. A radiator is composed of many such tube-fin assemblies arranged in series and parallel circuits. The radiating fin frequently also serves as the external structural skin of the vehicle. A complete analysis

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